The shape of the front of multidimensional branching Brownian motion

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Based on joint works with Julien Berestycki, Bastien Mallein, Eyal Lubetzky, and Ofer Zeitouni.

Branching Brownian motion (BBM)

Fix the dimension $d \geq 1$.

- Start: a single particle v at 0 performs Brownian motion in $\mathbb{R}^d$ (iid 1d BM's in each coordinate)

$$B_s(v) = (B_s^1(v), \dots, B_s^d(v)) \in \mathbb{R}^d$$

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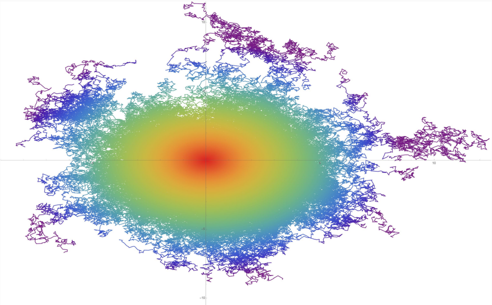

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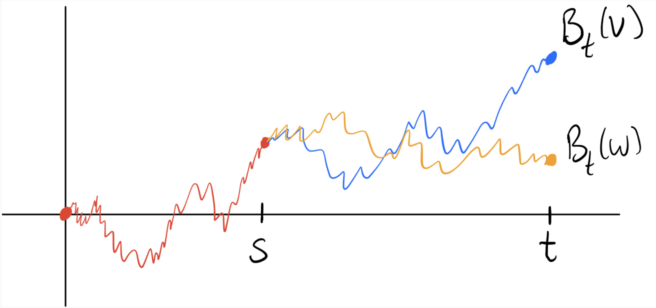
- $\mathbb{E}|N_t| = e^t$
- For each coordinate index i, $\text{Cov}(B_t^i(v), B_t^i(w)) =$ branching time of v and w

Covariance computation



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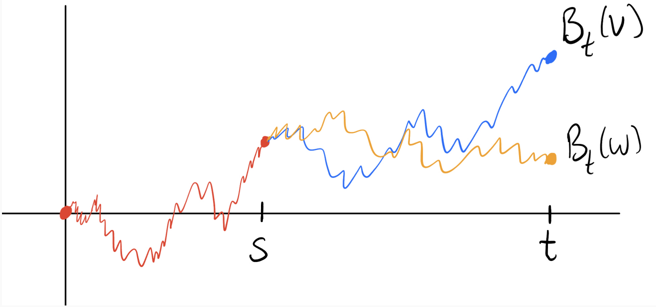

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$$\mathbb{E}\big[B_t(v)B_t(w) \mid \text{splitting times}\big]$$

$$= \mathbb{E}\Big[\Big(B_t(v) - B_s(v) + B_s(v)\Big)\Big(B_t(w) - B_s(w) + B_s(v)\Big) \mid \text{spl. times}\Big]$$

$$= \mathbb{E}\big[B_s(v)^2 \mid \text{spl. times}\big]$$

$$= s$$

BBM is a “log-correlated field”

- Consider some time $t \geq 0$, and consider some particle $v \in N_t$.
- Consider $\mathcal{B}(v, k) := \{w \in N_t : \mathrm{Cov}(B_t(v), B_t(w)) \geq k\}$.

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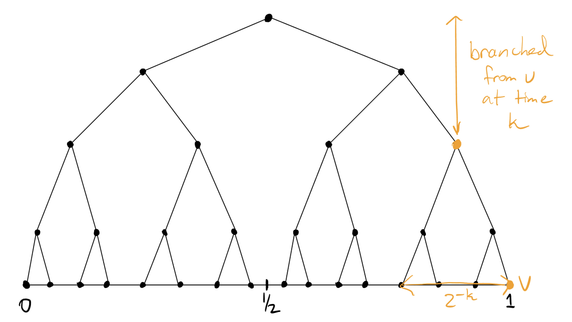

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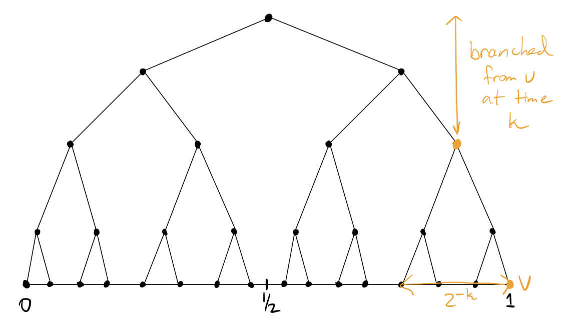

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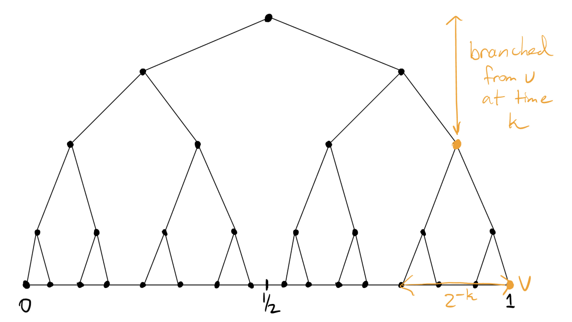

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$$\Rightarrow Cov(B_t(v), B_t(w)) \approx \log_2\left(\frac{1}{d(v,w)}\right).$$

Log-correlated fields: (discrete) Gaussian free field

Let $V_N := [1,N]^2 \cap \mathbb{Z}^2$. The discrete Gaussian free field on V_N (w/ 0 boundary conditions) is the field $\{h_v^{V_N} : v \in \mathbb{Z}^2\}$ with joint law

$$\mathrm{d}h^{V_N} := \frac{1}{Z} e^{-\frac{1}{8}\sum_{v\sim w}(h_v^{V_N}-h_w^{V_N})^2} \prod_{v\in V_N} \mathrm{d}h_v^{V_N} \prod_{v\notin V_N} \delta_0(\mathrm{d}h_v^{V_N}) \,.$$

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Properties:

- $\{h_v^{V_N}\}_{v\in\mathbb{Z}^2}$ is a Gaussian vector.
- $\mathrm{Cov}(h_v^{V_N}, h_w^{V_N}) = \frac{2}{\pi}\log\frac{N}{\max(\|v-w\|,1)} + O(\|v-w\|^{-2})$.

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$$= \operatorname{Re} \sum_{j=1}^{N} \sum_{k \geq 1} -\frac{e^{ik(\lambda_j - \theta)}}{k} = \operatorname{Re} \sum_{k \geq 1} -\frac{\operatorname{Tr} U_N^k}{k} e^{-ik\theta}$$

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Then, using the result of Diaconis-Shahshahani:

$$\mathrm{Cov}(X_N(\theta), X_N(\theta')) \asymp \mathbb{E} \sum_{k \geq 1} \frac{e^{-ik(\theta - \theta')}}{k^2} \mathrm{Re}\big[|\mathrm{Tr} U_N^k|^2\big]$$

$$\asymp \mathrm{Re} \sum_{k \geq 1} \frac{e^{-ik(\theta - \theta')}}{k} = -\mathrm{Re} \log(1 - e^{i(\theta - \theta')}) \asymp \log|\theta - \theta'|\,.$$

Other examples of LCFs

Examples come from...

- Random matrices (log-characteristic polynomials of beta-ensembles, Wigner, Ginibre,...)
- Interface models ($\nabla\phi$/Ginzburg-Landau models)
- Stochastic processes (local time of 2D Brownian motion, cover times of graphs)
- Even number theory (Riemann zeta function on the critical line, restricted to intervals of length 1)
- ...

Universality: extrema of log-correlated fields (predictions)

The *extreme values* of log-correlated fields are expected to exhibit universal behavior.

1. Maximum

   $\text{Max} \sim \textit{Leading order} - C \log(\textit{Leading order}) + \text{Gumbel} + \text{random shift}$

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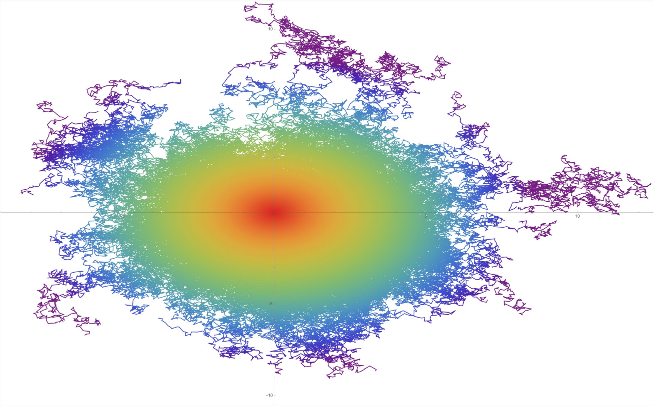
Finding a connection with BBM/BRW is crucial.

The Question of the Day



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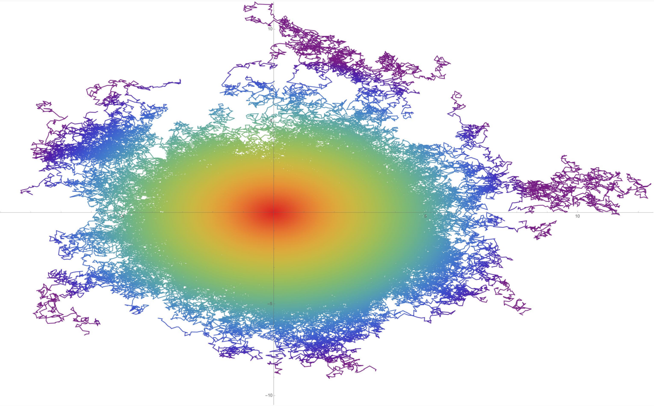
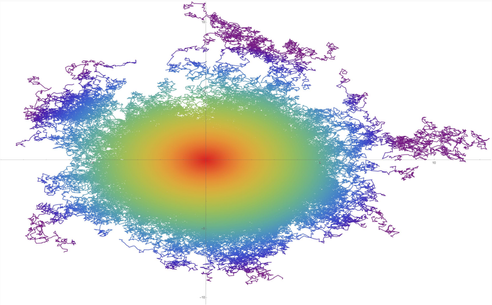


Question of the day: Simulate a BBM, run until time t. "Trace out" the outer edge of the picture formed by the BBM particles in N_t (the "front" of the BBM process). What is its shape?

The Question of the Day, take 2



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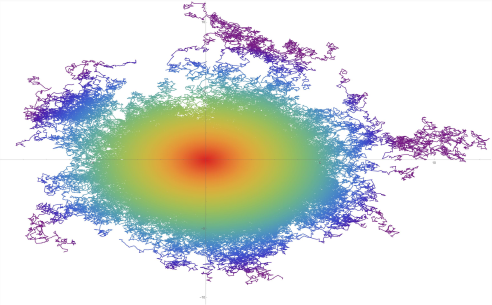

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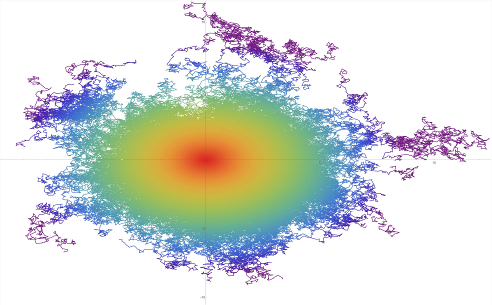

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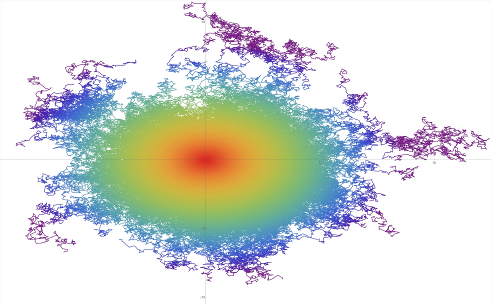

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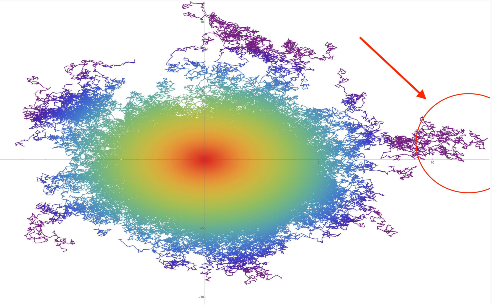

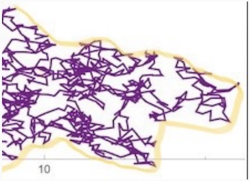

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The front and extreme value theory of BBM

Some definitions.

- For a particle $v \in N_t$, write $R_t(v) := \|B_t(v)\|$.
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So, the study of the front is tied to the "extremal landscape" of BBM.

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- Convergence in distribution was proved by Bramson '83 via connection with F-KPP equation, a reaction-diffusion equation
- identification of the limiting law as a Gumbel + random shift was proved by Lalley-Sellke '87

Why Gumbel + random shift?

The random shift

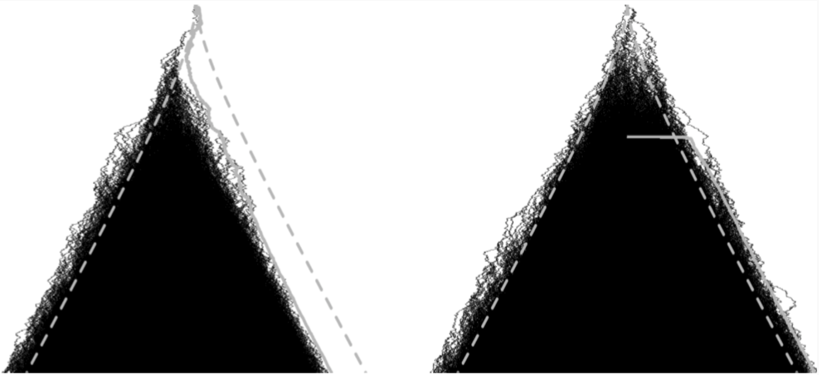

Figure 1: Left: initial particles veer to the left. Right: initial particles veer far to the right.

In both pictures, we see how the initial behavior permanently shifts the maximum. (Image by É. Brunet, taken from notes of J. Berestycki).

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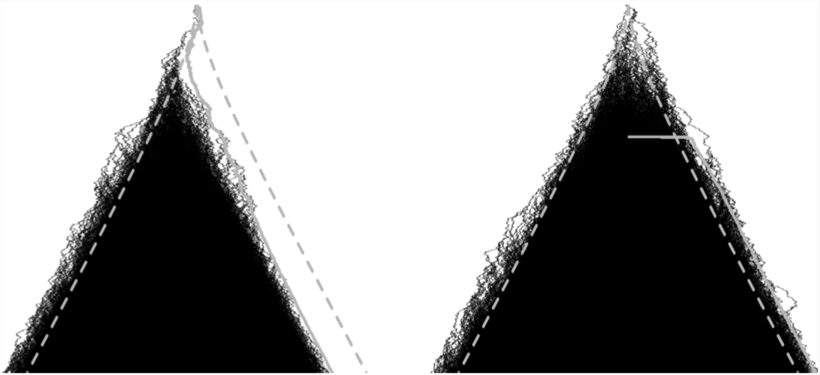

Theorem (Lalley-Sellke, '87)

$$\lim_{L\to\infty}\lim_{t\to\infty}\mathbb{P}\big(R_t^* - m_t(1) \le y \mid \mathcal{F}_L\big) \to \exp\big(-e^{-\sqrt{2}y+\log Z_\infty}\big) \; a.s.$$

Extremal landscape, dimension 1: Extremal point process

Consider the *extremal point process*

$$\mathcal{E}_t := \sum_{v \in N_t} \delta_{R_t(v) - m_t(1)} \, .$$

This is the point process of all particles near the maximum.

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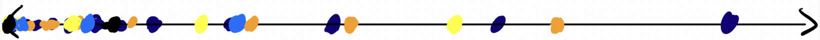

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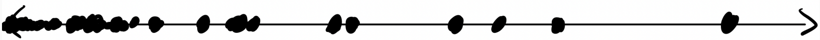

Randomly-shifted, decorated Poisson point process

Q: Where does the decorated Poisson point process come from?

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Q: Where does the decorated Poisson point process come from? Using a (modified) first and second moment method, one can show, for any $K \in \mathbb{R}$:

$$\lim_{L,\ell \to \infty} \lim_{t \to \infty} \mathbb{P}\Big(\exists v, w \in N_t : R_t(v) > m_t(1) - K, \\ R_t(w) > m_t(1) - K, MRCA(v, w) \in [L, t - \ell]\Big) = 0 \,.$$

This means all particles contributing to $\mathcal{E}_t$ are either very close relatives (branched after time $t - \ell$, gives the decoration) or extremely distant relatives (branched before time L, essentially independent, gives the PPP).

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  2. There's a new spatial aspect— the angles in addition to the norms of the particles.

     Fairly large gap in time between the 1d and multi-d results.

Extremal landscape, $\mathbb{R}^d$: Maximum

Previous results on the maximum of multidimensional BBM ($d \geq 2$).

1. (Leading-order term, Biggins '95): $R_t^*/\sqrt{2}t \to 1$ a.s.
2. (Sub-leading-order term and tightness, Mallein '15): $m_t(d) = \sqrt{2}t + \frac{d-4}{2\sqrt{2}} \log t$, and $(R_t^* - m_t(d))_{t \geq 0}$ is tight

Theorem (K.-Lubetzky-Zeitouni, Ann. Appl. Prob. '23)

There exists a a.s.-positive random variable Z_∞ such that

$$R_t^* - m_t(d) \Rightarrow \textit{Gumbel} - \frac{1}{\sqrt{2}} \log Z_\infty \, .$$

Connection with the F-KPP Equation

Consider the F-KPP reaction-diffusion equation:

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u + u(1-u) & \text{in } \mathbb{R}^d \\ u(0,x) = \phi(x) & \text{for } x \in \mathbb{R}^d \end{cases}$$

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- Equivalently, the median of the maximum norm of BBM in any fixed strip of width 1 is $m_t^G(d)$.
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Question: PDE approach to the maximum of multi-d BBM?

Extremal landscape, $\mathbb{R}^d$: Maximum, Proof Strategy

- Norm of Brownian motion in $\mathbb{R}^d$ is a d-dimensional Bessel process R.

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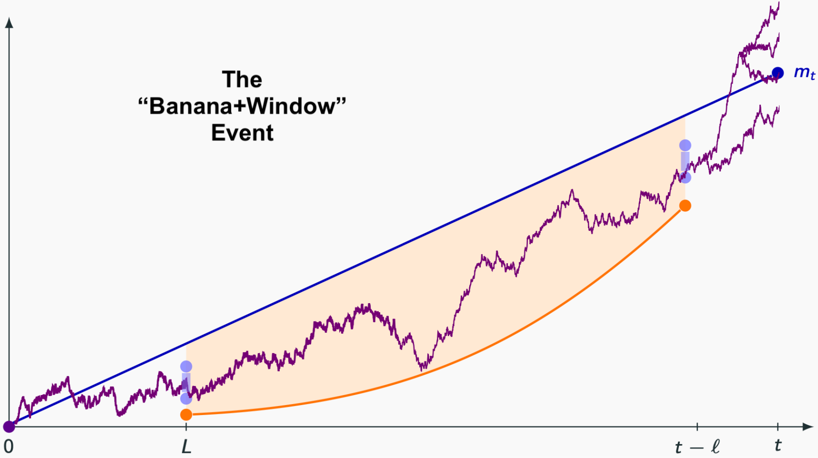
- We study the d-dim. branching Bessel process $\{R_s(v)\}_{s>0, v\in N_s}$.
- **Girsanov transform** gives Radon-Nikodym derivative with a 1d Brownian motion on an interval of time $[0, t]$:

$$\mathrm{d}P^R\big|_{\mathcal{F}_t} = \underbrace{\left(\frac{W_t}{W_0}\right)^{\frac{d-1}{2}}}_{\substack{\text{start/endpoint} \\ \text{dependence}}} \underbrace{\exp\left(\int_0^t \frac{c_d}{W_u^2}\mathrm{d}u\right)\mathbb{1}_{\{W_u>0,\ u\in[0,t]\}}}_{\text{pathwise dependence}} \mathrm{d}P^W\big|_{\mathcal{F}_t},$$

where $c_d > 0$ for $d \geq 3$ and $c_d < 0$ for $d = 2$.

Trajectories of the extremal particles

- Let L, ℓ be parameters that we send to infinity after t (think: constants wrt t).



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Consider the extremal point process.

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Key question: how does the early history of the process affect the distribution of the angles at later times, if at all?

Extremal landscape in $\mathbb{R}^d$: the extremal point process

Let L be a parameter going to infinity *after* $t \to \infty$ (so, with respect to t, L is a large but fixed constant).

Claim. If $v \in N_t$ is such that $R_t(v) > m_t(d)$, then $\|\theta_t(v) - \theta_L(v)\| = o(1)$ with high probability, where $o(1) \to 0$ after first $t \to \infty$ then $L \to \infty$.

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In words, the angle of an extremal particle at time t does not change after time L, where $L = O_t(1)$.

Proof of claim (angles of extremals freeze after $O(1)$ time)

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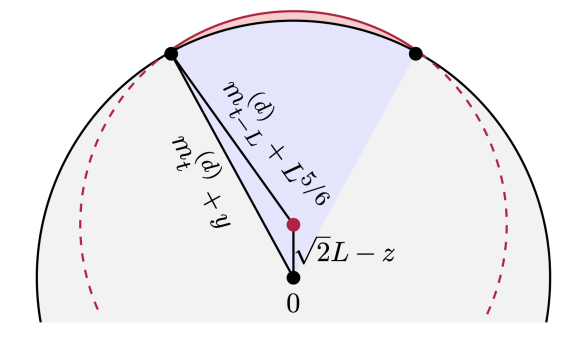
Proof Let $z := \sqrt{2}L - R_L(v) \asymp \sqrt{L}$ (we know this from the "window" event)



+exponential tail bounds on the max. displacement of BBM in $\mathbb{R}^d$.

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  - $D_\infty(\theta)$ is only defined as a function Leb-a.e., but still makes for a perfectly good density.
- (Aside) We later proved that the random shift Z_∞ of the $d \geq 2$ BBM maximum is given by the total mass $D_\infty(\mathbb{S}^{d-1})$.

Extremal landscape in $\mathbb{R}^d$: the extremal point process

Recall the extremal point process $\mathcal{E}_t := \sum_{v \in N_t} \delta_{(R_t(v) - m_t(d), \theta_t(v))}$

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Theorem (Berestycki-K.-L.-Mallein-Z., Ann. Prob '24+)

- *Let* $(\chi_i, \theta_i)_{i \in \mathbb{N}} \subset \mathbb{R}_+ \times \mathbb{S}^{d-1}$ *be the points of a*

$$\mathrm{PPP}\big(C_d e^{-\sqrt{2}x} \mathrm{d}x \times D_\infty(\theta) \mathrm{Leb}(d\theta)\big)\,,$$

  for some constant $C_d > 0$.

- *Let* $\{\mathcal{D}^{(i)}\}_{i \in \mathbb{N}}$ *be a collection of iid point processes with the same law as the decorations from the 1D BBM case.*

Then (weakly in the topology of vague convergence)

$$\mathcal{E}_t \to \mathcal{E}_\infty := \sum_{i \in \mathbb{N}} \sum_{r \in \mathcal{D}^{(i)}} \delta_{(\chi_i + r, \theta_i)}\,.$$

Extremal landscape in $\mathbb{R}^d$: the angular decorations

Theorem from last slide: Recall $\mathcal{E}_t := \sum_{v \in N_t} \delta_{(R_t(v) - m_t(d), \theta_t(v))}$.

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Q: Why are there no angular decorations?

A: We measure angles from the origin, but the clusters have diameter $O(1) \implies$ in the limit, the different angles in a cluster all get squashed.

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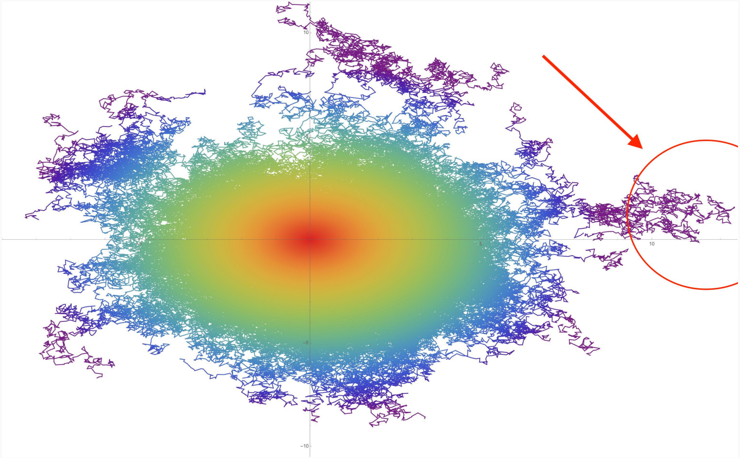

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$\implies$ the transversal motion of each particle in the cluster **after** time $t - O(1) \approx d - 1$ dimensional BM's, independent after conditioning on the genealogical tree.

Extremal BBM landscape: the angular decorations

Theorem (K., Zeitouni '24, Description of the extremal cluster)

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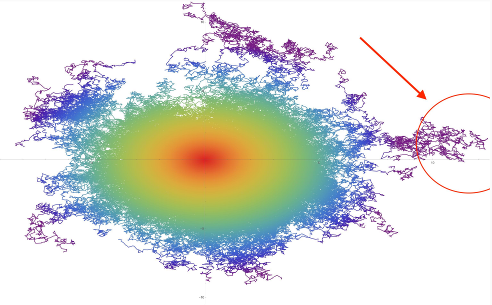
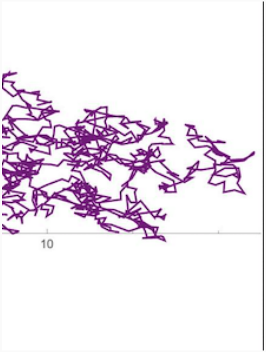
where $\mathcal{E}_\infty^{cluster}$ *is described below.*





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- For simplicity, let's focus on $d = 2$.

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2. (Branching times) At random branching times $0 < \tau_1 < \tau_2 < \ldots$ ($\approx$ PPP($2dt$)), the particle ξ produces a 2D BBM, started from $\mathcal{S}_{\tau_i}(\xi)$, run for time τ_i.

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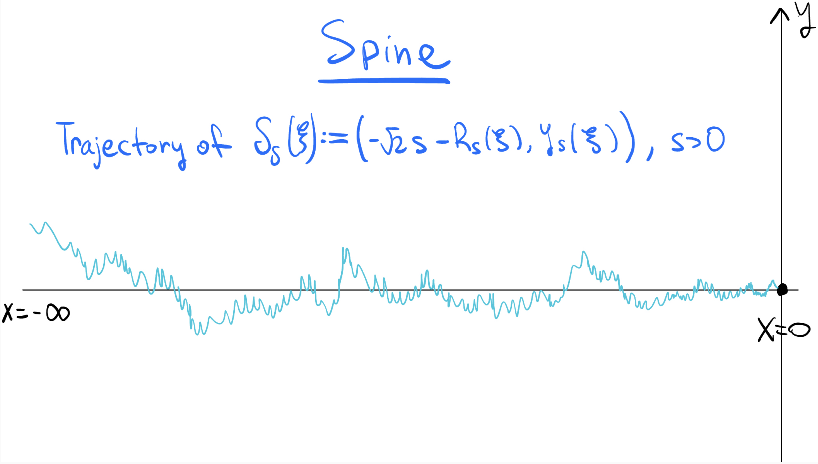

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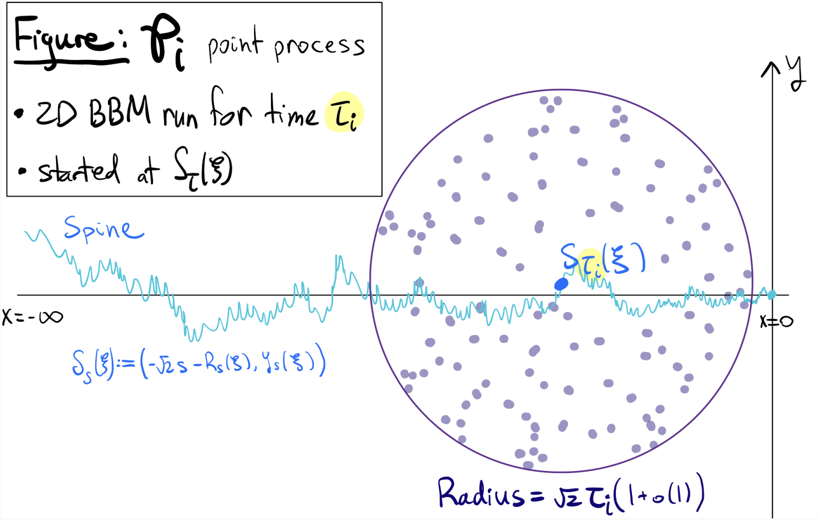

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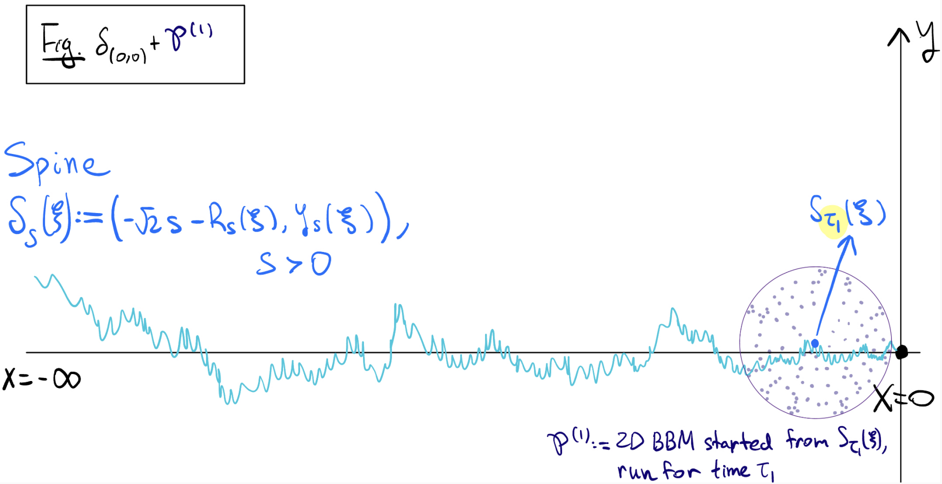

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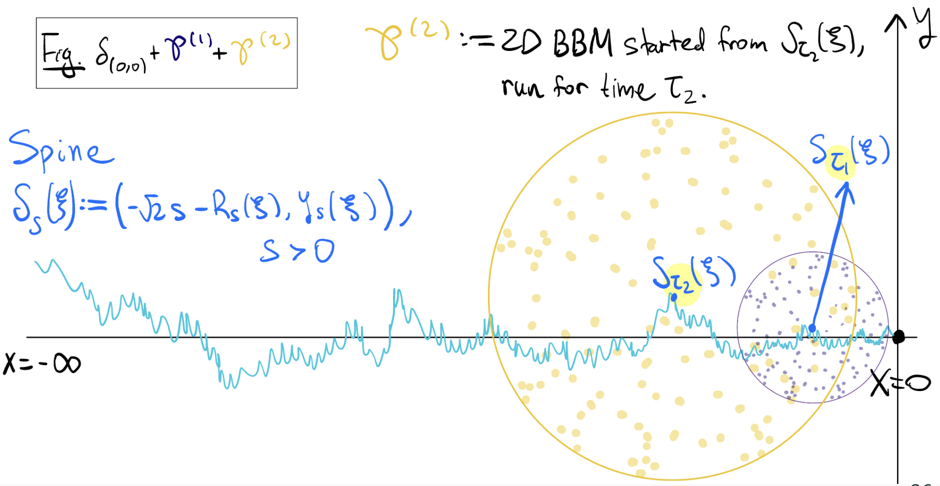

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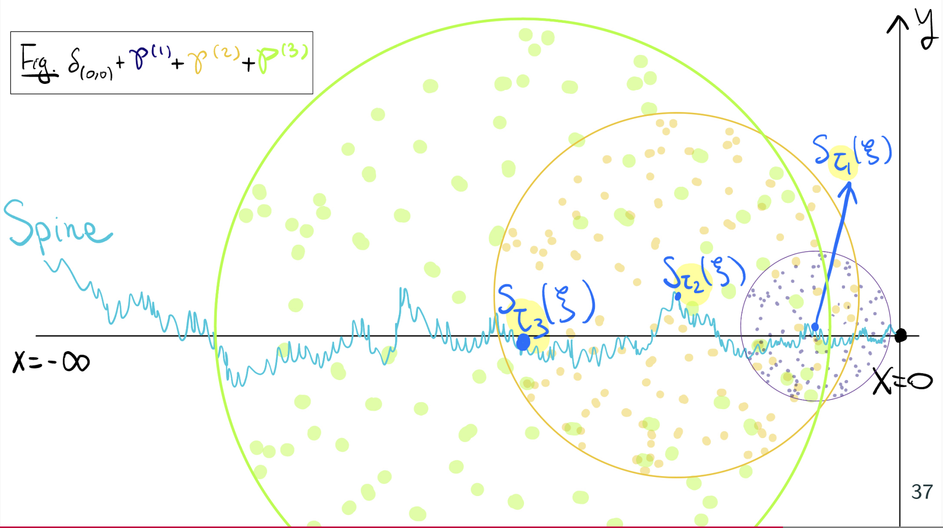

The Front of BBM

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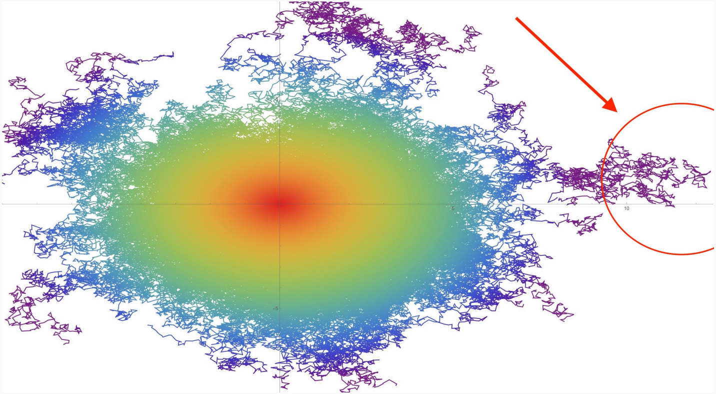

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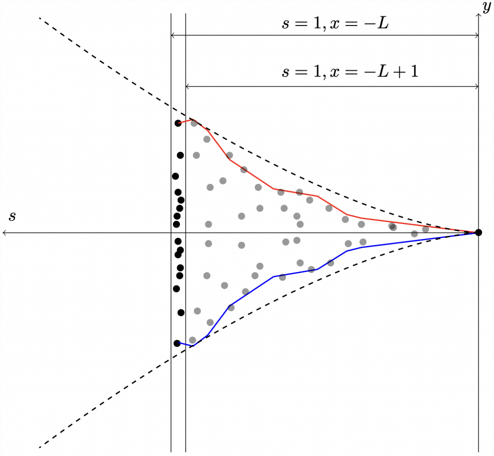

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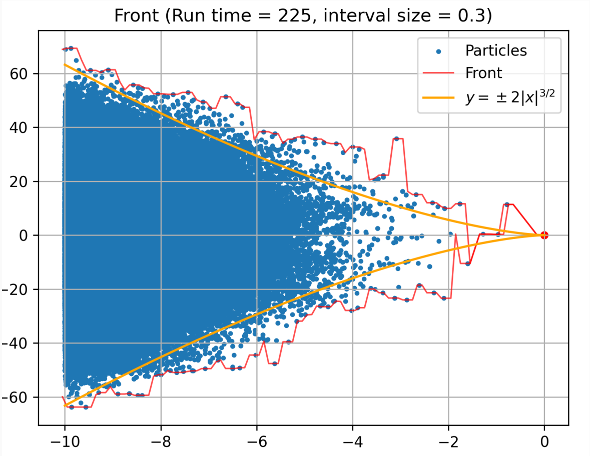

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Theorem (K., Zeitouni '24)

We have the following weak convergence of the front

$$\left(8^{-\frac{1}{4}} L^{-\frac{3}{2}} h_{t,L}(s)\right)_{s\in[0,\infty)} \Rightarrow (\rho_s)_{s\in[0,\infty)}\,,$$

as first $t \to \infty$, then $L \to \infty$, where

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For any $d \geq 2$, the front converges to the paraboloid formed by rotating $\rho_\cdot$ around the x-axis.

Heuristic argument for the $3/2$ scaling exponent, $d = 2$

Let's understand the $L^{3/2}$ behavior of $h_{t,L}(s)$, say for $s = 1$, $d = 2$.

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- Due to weak convergence of $\mathcal{E}_t^{cluster}$ to the explicit point process $\mathcal{E}_\infty^{cluster}$ as $t \to \infty$, it suffices to study the front of $\mathcal{E}_\infty^{cluster}$:

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- There will be an exponential number of particles (in L) in this strip $\rightarrow$ we'll ignore polynomial terms.
- Also, to leading-order, the max. of log-correlated fields agrees with the max. of iid fields $\rightarrow$ we'll pretend the particle trajectories are *independent* Brownian motions.

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Solve: $e^{\mathbf{M}_\tau^2/2\tau} = e^{\mathbf{N}} \implies \mathbf{M}_\tau^2 = 2\Big(-\alpha_\tau\sqrt{2}\tau^{3/2} + \sqrt{2}L\tau + \alpha_\tau L\tau^{1/2} - \frac{L^2}{2}\Big)$

Treating α_τ as constant and optimizing over τ yields $\tau \asymp L^2$, and thus $\mathbf{M}_\tau \asymp \tau^{3/2}$. Write $\tau = \sigma L^2$:

$$\mathbf{M}_s^2 \approx 2\sqrt{2} L^3 \Big(\sigma - \sigma \frac{R_{\sigma L^2}(\xi)}{L}\Big) \Rightarrow h_L(1) \approx 8^{\frac{1}{4}} L^{\frac{3}{2}} \max_{\sigma \geq 0} \Big(\sigma - \sigma \frac{R_{\sigma L^2}(\xi)}{L}\Big)^{1/2}.$$

Glimpse of the rigorous argument

$$h_L(s) = \max_{i \in \mathbb{N}} \max_{i^{th} \text{ BBM cloud}} \left\{ \text{vertical displacement in } [-sL, -sL+1] \times \infty \right\}$$

Glimpse of the rigorous argument

$$h_L(s) = \max_{i\in\mathbb{N}} \max_{i^{th}\text{ BBM cloud}} \left\{\text{vertical displacement in } [-sL, -sL+1] \times \infty\right\}$$

- Avoids any modified second moment method that has become standard in the study of log-correlated fields.
- Proceeds by
    1. Localizing the set of birth times of the BBM clouds which contribute to $L^{-3/2}h_L(s)$
    2. Understanding how much space is filled by each of these BBM clouds

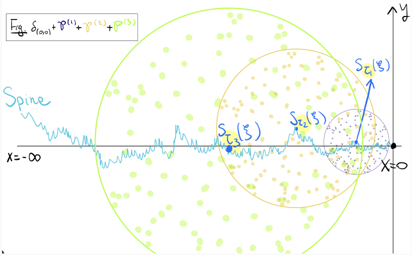
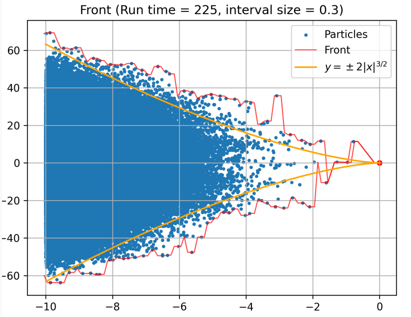


Fig. $\delta_{(0,0)} + \mathcal{P}^{(1)} + \mathcal{P}^{(2)} + \mathcal{P}^{(3)}$
Spine
y
$S_{\tau_1}(\xi)$
$S_{\tau_2}(\xi)$
$S_{\tau_3}(\xi)$
$X=-\infty$
$X=0$


Front (Run time = 225, interval size = 0.3)
Particles
Front
$y = \pm 2|x|^{3/2}$
60
40
20
0
-20
-40
-60
−10
−8
−6
−4
−2
0

Thank you!