Absolute continuity of non-Gaussian and Gaussian multiplicative chaos measures

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Institut Mittag-Leffler, Dec. 2024

Based on joint work with Xaver Kriechbaum.

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- We'll answer through the lens of absolute continuity.

Outline

1. Background on log-correlated fields and Gaussian multiplicative chaos
2. Multiplicative chaos from non-Gaussian log-correlated fields
3. Main theorem
4. Proof ideas with a view towards future work

Log-correlated fields

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Examples.

- Random matrices (log-characteristic polynomials of beta-ensembles, Wigner, Ginibre,. . .)
- Interface models (Gaussian free field, $\nabla\phi$/Ginzburg-Landau)
- Stochastic processes (branching Brownian motion/random walk, local time of 2D Brownian motion, cover times of graphs)
- Even number theory (Riemann zeta function on the critical line, restricted to intervals of length 1)
- ...

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$$X_N(\theta) := \sum_{j=1}^{N} \log \left| 1 - e^{2\pi i(\lambda_j - \theta)} \right|$$

$$= \operatorname{Re} \sum_{j=1}^{N} \sum_{k \geq 1} -\frac{e^{2\pi ik(\lambda_j - \theta)}}{k} = \operatorname{Re} \sum_{k \geq 1} -\frac{\mathrm{Tr}\, U_N^k}{k} e^{-2\pi ik\theta}$$

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$$\mathrm{Cov}(X_N(\theta), X_N(\theta')) \asymp \mathrm{Re} \sum_{k \geq 1} \frac{e^{-2\pi i k(\theta - \theta')}}{k^2} \mathbb{E}\left[|\mathrm{Tr} U_N^k|^2 \right]$$

$$\asymp \mathrm{Re} \sum_{k \geq 1} \frac{e^{-2\pi i k(\theta - \theta')}}{k} = -\mathrm{Re} \log(1 - e^{2\pi i(\theta - \theta')}) \asymp -\log |\theta - \theta'| \,.$$

Example in $d = 2$: (discrete) Gaussian free field

Let $V_N := [1, N]^2 \cap \mathbb{Z}^2$. The discrete Gaussian free field on V_N (w/ 0 boundary conditions) is the field $\{h_v^{V_N} : v \in \mathbb{Z}^2\}$ with joint law

$$\mathrm{d}h^{V_N} := \frac{1}{Z} e^{-\frac{1}{8}\sum_{v\sim w}(h_v^{V_N} - h_w^{V_N})^2} \prod_{v\in V_N} \mathrm{d}h_v^{V_N} \prod_{v\notin V_N} \delta_0(\mathrm{d}h_v^{V_N}) \,.$$

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Properties:

- $\{h_v^{V_N}\}_{v\in\mathbb{Z}^2}$ is a Gaussian vector.
- $\mathrm{Cov}(h_v^{V_N}, h_w^{V_N}) = \frac{2}{\pi} \log \frac{N}{\max(\|v-w\|,1)} + O(\|v - w\|^{-2})$.

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1. $(d = 1)$ **CUE**. The log-characteristic poly. $(X_n(\theta))_{\theta \in [0,1]}$ of CUE converges to the log-correlated random Fourier series [Hughes-Keating-O'Connell '01]

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2. $(d = 2)$ **Ginibre**. The log-char. poly $(X_n(z))_{z \in \mathbb{D}}$ of a Ginibre matrix (i.i.d. complex Gaussian entries) converges to [Rider-Virag '07]

$$X_N(z) \to \text{Gaussian free field in } \mathbb{D}\,.$$

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$$\nu_{N,\beta}(\mathrm{d}x) := \frac{\mathbb{1}_{\{x \in D_N \,:\, X_N(x) \geq \beta \max_{y \in D_N} X_N(y)\}}}{\mathbb{P}(x \in D_N : X_N(x) \geq \beta \max_{y \in D_N} X_N(y))} \, \mathrm{d}x$$

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- extreme level sets of **BBM** [Genz-Kistler-Schmidt '18]
- extreme level sets of **DGFF** [Biskup-Louidor '19]
- extreme level sets of **CUE**, Gaussian LCFs [Junnila-Lambert Webb '24]
- LCP of **CUE**: [Webb '15] (L^2-phase), [Nikula-Saksman-Webb '20] (full subcritical)
- dynamics on LCP of **CUE** [Bourgade-Falconet '22]
- spectral measure of **C**β**E** [Chhaibi-Najnudel '19]
- LCP of **C**β**E** [Lambert-Najnudel '24]
- LCP of **GUE** [Berestycki-Webb-Wong '18]
- Eigenvalue counting function of **GUE** [Claeys-Fahs-Lambert-Webb '21]
- LCP of **GOE/GSE** [Kivamae '24]
- ...

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- h can be realized as a Gaussian random variable in the Sobolev space $H^s(\mathbb{R}^d)$, for all $s < 0$.

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- Original motivations come from quantum field thoery [Høeg-Krohn '71] and turbulance [Mandelbrodt '85]. Further applications (beyond what's been mentioned) include 2D Liouville quantum gravity, finance.

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3. Riemann zeta function
    - Non-Gaussianity comes from contribution of small primes.
    - [Saksman-Webb '20] showed, for a random model of zeta, the corresponding multiplicative chaos is absolutely continuous w.r.t. a coupled GMC, with bounded R-N derivative.

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   - Another paper [Caravenna-Sun-Zygouras '23]: "The (one-time marginal of) critical 2d SHF is not a GMC"
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5. Brownian multiplicative chaos
   - Multiplicative chaos measure coming from the field of local times of 2D Brownian motion
   - Closure of support is the trajectory of a 2D Brownian motion
   - Originally constructed in L^2-regime by [Bass-Burdzy-Khoshnevisan '94], in whole subcritical regime by [Aïdékon-Hu-Shi '20] and [Jego '20]
     - General characterization by [Jego '23], critical case by [Jego '21], connection with Brownian loop soup by [Aïdékon-Berestycki-Jego-Lupu '23]

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So, convergence as $n \to \infty$ of $\mu_{n,\gamma,a}$ is easy. Non-triviality of limit?

Multiplicative Chaos of non-Gaussian Fourier series

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Question: Does $\mu_{\gamma,a}$ resemble a GMC_γ in any way?

Main Result

Theorem (K.-Kriechbaum, '24)

Take $\gamma \in (1, \sqrt{2})$. There exists a sequence of i.i.d. standard Gaussians $g := (g_k^{(i)})_{k \in \mathbb{N}, i \in \{1,2\}}$ coupled to $a := (a_k^{(i)})$ such that $\mu_{\gamma,g}$ and $\mu_{\gamma,a}$ are mutually absolutely continuous, a.s.

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  - **Remark.** Result seems to improve as more moments of $a_k^{(i)}$ match those of a Gaussian (e.g. $\mathbb{E}[(a_k^{(i)})^3] = 0$)

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- But robust: insensitive to choice of γ and domain D.

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  - Caveat: $(\tilde{a}_m(x))_{x\in[0,1]}$ are **not independent** across different x's

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Claim: For each $m \in \mathbb{N}$, $\widetilde{a}_m(x)$ is piecewise constant on intervals of length $\ll 2^{-m}$.

Heuristic proof: For $x, x' \in [0,1]$,

$$\widetilde{a}_m(x) - \widetilde{a}_m(x') = \sum_{k=2^{m-1}}^{2^m-1} \frac{a_k^{(1)}}{\sqrt{k}} \Big(\cos(2\pi kx) - \cos(2\pi kx') \Big) + \ldots \sin \ldots$$

$$\asymp \underbrace{\frac{1}{\sqrt{2^{m-1}}} \sum_{k=2^{m-1}}^{2^m-1}}_{\text{By CLT, } = O(1)} a_k^{(1)} \underbrace{\Big(\cos(2\pi kx) - \cos(2\pi kx') \Big) + \ldots \sin \ldots}_{\text{By Lipschitz cont., } = O(k \cdot |x-x'|) = O(2^m |x-x'|)}$$

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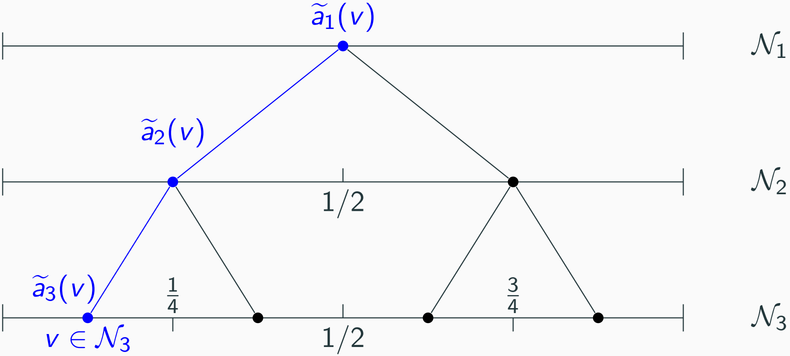

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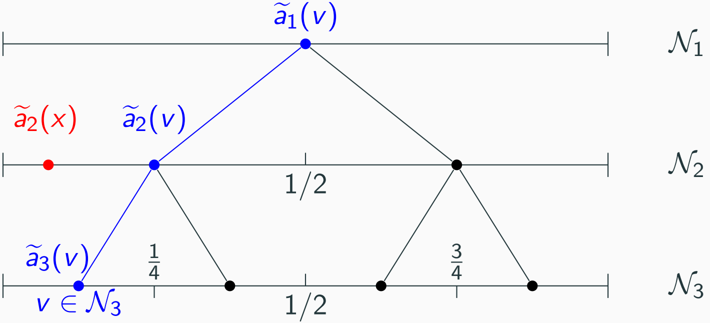

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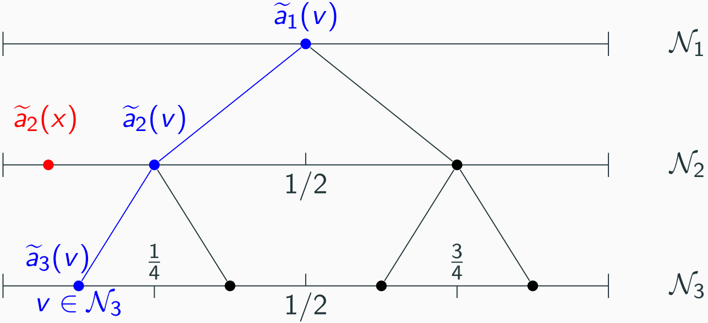

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and the piecewise constant field

$$\widetilde{S}_{n,a}(x) := \widetilde{S}_{n,a}(v) = \sum_{m=1}^{n} \widetilde{a}_m(v)$$

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- Thus, the chaos $\widetilde{\mu}_{\gamma,\widetilde{a}}$ asso'd to $\widetilde{S}_{n,a}$ is a.s. mutually absolutely continuous w.r.t. the chaos $\mu_{\gamma,a}$ asso'd to $S_{n,a}$:

$$\begin{aligned} \widetilde{\mu}_{\gamma,\widetilde{a}}(\mathrm{d}x) &:= \lim_{n\to\infty} \frac{e^{\gamma \widetilde{S}_{n,a}(x)}}{\mathbb{E}[e^{\gamma \widetilde{S}_{n,a}(x)}]}\mathrm{d}x \\ &= \lim_{n\to\infty} \underbrace{e^{\gamma(S_{n,a}(x) - \widetilde{S}_{n,a}(x))} \cdot \frac{\mathbb{E}[e^{\gamma S_{n,a}(x)}]}{\mathbb{E}[e^{\gamma \widetilde{S}_{n,a}(x)}]}}_{\text{uniformly bounded, a.s.}} \cdot \underbrace{\frac{e^{\gamma S_{n,a}(x)}}{\mathbb{E}[e^{\gamma S_{n,a}(x)}]}\mathrm{d}x}_{\mu_{n,\gamma,a}\to\mu_{\gamma,a}} \end{aligned}$$

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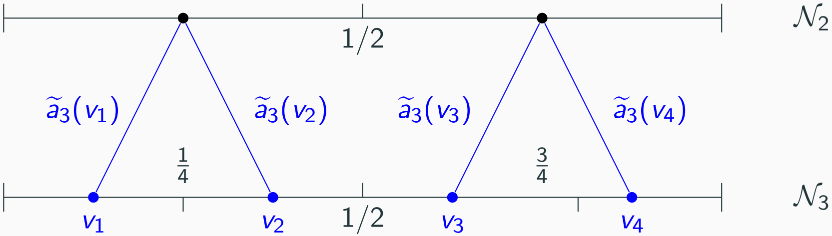

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Fix $M, D \in \mathbb{N}$. Let $\xi_1, \dots, \xi_M$ be independent, centered random $\mathbb{R}^D$-vectors. Let $\vec{a} := \sum_{k=1}^M \xi_k$. Then $\exists \vec{g} \sim N(0, \mathrm{Cov}(\vec{a}))$ coupled to $\vec{a}$ such that

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for any $p \in [1, \infty]$.

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Yurinskii coupling for us

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$$\xi_k := \Big(\frac{a_k^{(1)}\cos(2\pi kv) + a_k^{(2)}\sin(2\pi kv)\ldots}{\sqrt{k}}\Big)_{v\in\mathcal{D}}\ .$$

- Note ξ_k only has 2 sources of randomness: $a_k^{(1)}$ and $a_k^{(2)}$.
- Also, for the $\mathcal{D}$ we eventually choose, we can assume the cos and sin are bounded away from 0 for a positive proportion of the v's.

$$\|\xi_k\|_p \approx \frac{\max(|a_k^{(1)}|, |a_k^{(2)}|)}{k^{1/2}}\|(1,\ldots,1)\|_p \asymp 2^{-\frac{m}{2}}\max(|a_k^{(1)}|, |a_k^{(2)}|)|\mathcal{D}|^{\frac{1}{p}}\ .$$

Step 2 (Coupling step): Yurinskii coupling

Let's estimate the error term:

$$\mathbb{P}\Big(\|\vec{a} - \vec{g}\|_p > \delta\Big) \leq \frac{1}{\delta^3} \sum_{k=2^{m-1}}^{2^m - 1} \mathbb{E}\Big[\|\xi_k\|_2^2 \cdot \|\xi_k\|_p\Big] + \textit{stuff} \,,$$

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$$\xi_k := \Big(\frac{a_k^{(1)} \cos(2\pi k v) + a_k^{(2)} \sin(2\pi k v) \dots}{\sqrt{k}}\Big)_{v \in \mathcal{D}} \,.$$

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- Minimized when $p = \infty$!

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Fix $\mathcal{D} \subset \mathcal{N}_m$, and write the corresponding level-m increments as a sum of 2^{m-1} independent vectors in $\mathbb{R}^{|\mathcal{D}|}$:

$$\big(\widetilde{a}_m(v)\big)_{v\in\mathcal{D}} = \sum_{k=2^{m-1}}^{2^m-1} \underbrace{\left(\frac{a_k^{(1)}\cos(2\pi k v) + a_k^{(2)}\sin(2\pi k v)\dots}{\sqrt{k}}\right)_{v\in\mathcal{D}}}_{\xi_k}$$

Yurinskii coupling for us

Fix $M = 2^{m-1}, D = |\mathcal{D}|$. Let $\vec{a} := (\widetilde{a}_m(v))_{v\in\mathcal{D}} = \sum_{k=2^{m-1}}^{2^m-1} \xi_k$.

Then $\exists \vec{g} := (\widetilde{g}_m(v))_{v\in\mathcal{D}} \sim N(0, \mathrm{Cov}(\vec{a}))$ coupled to $\vec{a}$ such that

$$\mathbb{P}\Big(\|\vec{a} - \vec{g}\|_p > \delta\Big) \leq \frac{1}{\delta^3} \sum_{k=2^{m-1}}^{2^m-1} \mathbb{E}\Big[\|\xi_k\|_2^2 \cdot \|\xi_k\|_p\Big] + \textit{stuff}$$

for any $p \in [1, \infty]$.

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Q: What should $\mathcal{D} \subset \mathcal{N}_m$ be?

Step 2 (Coupling step): Thick points

- **Key fact:** (G)MC_γ is supported on the γ-thick points of the field

$$\mathcal{T} := \left\{ x \in [0,1] : \liminf_{n\to\infty} \frac{\widetilde{S}_{n,\widetilde{a}}(x)}{(\log 2)n} = \gamma \right\}, \quad \mathrm{MC}_\gamma(\mathcal{T}^c) = 0\,.$$

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- $\implies$ It should be sufficient to couple along $\mathcal{T}$.
- But $\mathcal{T}$ is defined in terms of a limit, and our coupling scheme goes level-by-level... want finite$-n$ version of thick points.

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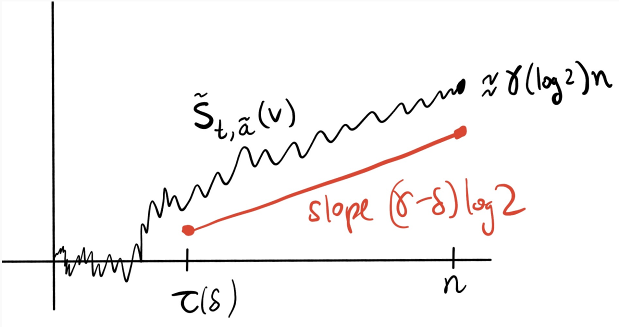

Figure 1: Trajectory of the walk at a thick point. After some random time $\tau(\delta) < \infty$ a.s., it should stay above $\ell(t)$.

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$$\mathcal{T}_{n,\gamma,\widetilde{a}} := \left\{ v \in \mathcal{N}_n : \widetilde{S}_{n,\widetilde{a}}(v) \geq \log(2)(\gamma-\delta)n \right\}$$

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- Define the random set of direct descendants of $\mathcal{T}_{n,\gamma}$:

$$\mathcal{N}_{n+1}(\mathcal{T}_{n,\gamma}) := \{v \in \mathcal{N}_{n+1} : \exists w \in \mathcal{T}_{n,\gamma} \text{ such that } v \text{ descends from } w\}\,.$$

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- Our Yurinskii coupling takes place on $(\widetilde{a}_{n+1}(v))_{v\in\mathcal{N}_{n+1}(\mathcal{T}_{n,\gamma})}$. Crucially, these values are still independent of levels 1 to n.

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Yurinskii: $\mathbb{P}\left(\|\vec{a} - \vec{g}\|_\infty > n^{-2}\right) \leq Cn^6 2^{-n/2} |\mathcal{D}|$

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- Can extend to $(\widetilde{g}_{n+1}(v))_{v\in\mathcal{N}_{n+1}}$ so that its covariance matches $(\widetilde{a}_{n+1}(v))_{v\in\mathcal{N}_{n+1}}$.

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$$\begin{array}{ccccl} \mu_{\gamma,a} & & \mu_{\gamma,g} & & \text{(chaoses from Fourier series)} \\ \updownarrow & & \updownarrow & & \\ \widetilde{\mu}_{\gamma,\widetilde{a}} & \overset{?}{\longleftarrow\!\!\!-\!\!\!-\!\!\!\longrightarrow} & \widetilde{\mu}_{\gamma,\widetilde{g}} & & \text{(chaoses from tree models)} \end{array}$$

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Recall the chaos measure

$$\widetilde{\mu}_{\gamma,\widetilde{a}}(A) := \lim_{n\to\infty} \int_A \frac{e^{\gamma \widetilde{S}_{n,\widetilde{a}}(x)}}{\mathbb{E}[e^{\gamma \widetilde{S}_{n,\widetilde{a}}(x)}]} \, \mathrm{d}x \,.$$

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$$R_{\infty,\gamma}(x) := \lim_{n\to\infty} R_{n,\gamma}(x)\,, \quad R_{n,\gamma}(x) := e^{\gamma(\widetilde{S}_{n,\widetilde{a}}(x) - \widetilde{S}_{n,\widetilde{g}}(x))} \frac{\mathbb{E}[e^{\gamma \widetilde{S}_{n,\widetilde{g}}(x)}]}{\mathbb{E}[e^{\gamma \widetilde{S}_{n,\widetilde{a}}(x)}]}$$

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Need to show:

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   - \+ Items 2 and 3 form the most technical part of paper.

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- L^∞-norm was **optimal** for Yurinskii's coupling
- (Cattaneo-Masini-Underwood '22): error in Yurinskii's coupling improves if $\mathbb{E}[(a_1^{(1)})^3] = 0$. Examining their proof seems to show the error continues to improve if more moments match Gaussian's.

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3. Properties of $R_{\infty,\gamma}$?

- $\sup_{x \in [0,1]} R_{\infty,\gamma}(x)$ is not a.s. finite.

Summary

Theorem (K.-Kriechbaum, arXiv:2410.19979)

Take $\gamma \in (1, \sqrt{2})$. The log-correlated random Fourier series with general coefficients can be coupled with a Gaussian Fourier series such that the associated multiplicative chaos measures are mutually absolutely continuous, a.s.

Future work.

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Tack!